Deriving the Formulas for Multi-facet Temari

Formulas for a C10 based multi-facet marking

6-part triangle method			4-part diamond method		
Number used to divide triangle side	facets	perfect facets after multiple of three method	Number used to divide diamond side	facets	perfect facets after multiple of three method
x	$\frac{1}{3} \cdot 10x^2 - 2$	$10x^2 + 2$	y	$10y^2 + 2$	$30y^2 + 2$

Although there are four formulas all together, we can consider them in two groups: those for perfectly split hexagons after applying the multiple of three method and those for imperfectly split hexagons after using either the triangle or diamond method. Let's start with the imperfect formulas.

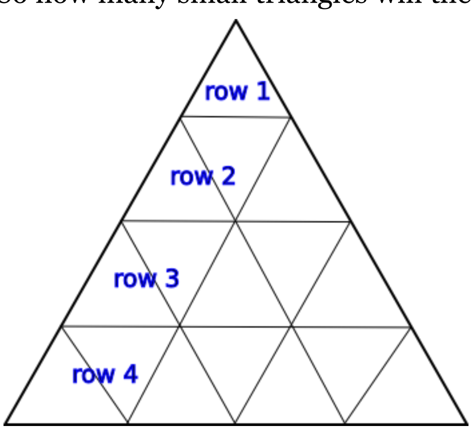
Imperfect Facets Formulas

In both the triangle and diamond method we are splitting the C10 into a number of smaller triangles. Those smaller triangles are recombined into pentagons and hexagons. Each pentagon formed will need 5 triangles. Since there are always 12 pentagons, they will use 12 x 5 = 60 small triangles. All of the other triangles will be combined to make hexagons. Each hexagon is made of 6 triangles so:

$$hexagons = \frac{small\ triangles - 60}{6}$$

Triangle Method

So how many small triangles will there be? Let's consider the triangle method first. In this case we split the sides of the large triangle equally forming a number of smaller triangles. When we split the sides of the triangle we create rows of smaller triangles. The first row will have 1, the second row will have 3, the third row will have 5 and so on up to the number of rows needed (x). That means we'll have 1 + 3 + 5 +. . . or the sum of the first x odd numbers, which is x^2. There is a neat visual proof of this, but this is not the place for that.



Now, there are 20 large triangles on the C10. They are equivalent to the faces of an icosahedron. So there are $20x^2$ small triangles. We can substitute that into the formula for hexagons and simplify:

$$hexagons = \frac{small\ triangles - 60}{6}$$

$$hexagons = \frac{20x^2 - 60}{6}$$

$$hexagons = \frac{20x^2}{6} - \frac{60}{6}$$

$$hexagons = \frac{10x^2}{3} - 10$$

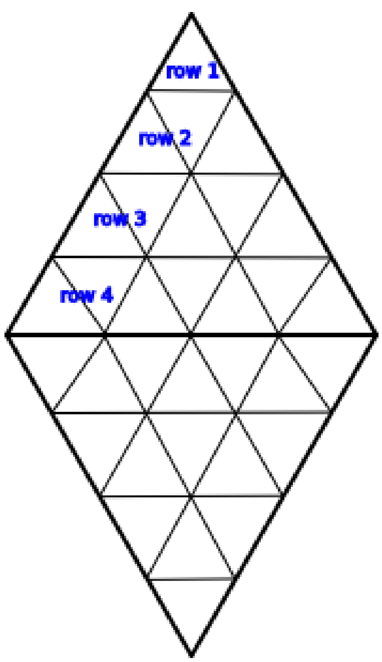
And the total number of facets is the hexagons plus the 12 pentagons or

$$facets = \frac{10x^2}{3} - 10 + 12$$

$$facets = \frac{1}{3} \cdot 10x^2 - 2$$

Diamond Method

The process is similar for the diamond method. The diamond can be considered to be two triangles placed together so the number of small triangles in the diamond with a side split into Y sections would be $2y^2$. There are 30 diamonds on a C10 so the total number of small triangles is $30 \cdot 2y^2$ or $60y^2$. So using our formula for the number of hexagons again:



$$hexagons = \frac{small\ triangles - 60}{6}$$

$$hexagons = \frac{60y^2 - 60}{6}$$

$$hexagons = \frac{60y^2}{6} - \frac{60}{6}$$

$$hexagons = 10y^2 - 10$$

And the total number of facets is the hexagons plus the 12 pentagons or

$$facets = 10y^2 - 10 + 12$$

$$facets = 10y^2 + 2$$

Perfect Formulas

For the perfect formulas we need to consider how the multiple of three method works. It creates a 12-way intersection at each existing vertex on the ball and creates new 6-way vertices at the middle of each triangle. That means that every existing vertex will become the center of a face. So to count the number of facets we need only count the number of vertices that are there to start with. Faces? Vertices? This sounds like a job for Euler. The mathematician Euler proved that there is a special relationship between the edges (E), faces (F) and vertices (V) of a polyhedron.

$$V = E - F + 2$$

We'll use this relationship to calculate the vertices on the mari when using either the triangle or diamond method.

Triangle Method

In the earlier section we showed that the number of small triangles is $20x^2$. Those are the faces for this part. Each of those faces is a triangle and has three edges. Each edge is shared by two faces so the number of edges is $\frac{20x^2 \cdot 3}{2}$. The vertices will be the number of facets in the marking. Let's substitute these into Euler's relationship:

$$facets = \frac{20x^2 \cdot 3}{2} - 20x^2 + 2$$

$$facets = 30x^2 - 20x^2 + 2$$

$$facets = 10x^2 + 2$$

Diamond Method

We derive the formula for the diamond method in exactly the same way. The number of faces is $60y^2$ faces. The number of edges will be 3 times that divided by 2 or:

$$\frac{60y^2 \cdot 3}{2} = 90y^2$$

Substituting into Euler's formula we get:

$$facets = 90y^2 - 60y^2 + 2$$

And simplify to:

$$facets = 30y^2 + 2$$

What about the C8 formulas?

The C8 formulas can be derived in the same way with a few small differences. First of all on C8 multipoles there are 6 squares made up of four small triangles each rather than the 12 pentagons made up of five small triangles. Second, the numbers of large triangles and diamonds are different. There are 8 large triangles (faces of an octahedron) and 12 diamonds on a C8. I'll leave it to you to do the actual derivation if you choose.